Epic math battle of history: Grothendieck vs Nikodym - Round 2

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Motivation and the main result

Theorem (Talagrand, 1984): Assume CH. Then there is a Boolean algebra with the Grothendieck property and without the Nikodym property.

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Theorem (G. & Widz)

There is a σ-centered (and so ccc) notion of forcing $\mathbb{P}$ such that

$$\mathbb{P} \Vdash \quad \text{there exists a Boolean algebra of cardinality } \omega_1 \text{ with the Grothendieck property and without the Nikodym property}$$

In particular, the existence of such an algebra is consistent with $\neg$CH.

Normal sequences

We say that a sequence of (finitely additive bounded signed) measures $(\nu)_{n\in\mathbb{N}}$ on a Boolean algebra $\mathbb{B}$ is **normal** if

- $\forall n \in \mathbb{N}\ \|\nu_n\| = 1$,
- the Radon measures $\widetilde{\nu}_n$ on $\mathrm{St}(\mathbb{B})$ extending ν_n are concentrated on pairwise disjoint Borel sets.

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Fact

If $\mathbb{B}$ does not have the Grothendieck property, then there is a normal sequence of measures $(\nu_n)_{n\in\mathbb{N}}$ on $\mathbb{B}$ such that $(\widetilde{\nu}_n)_{n\in\mathbb{N}}$ converges in the weak*-topology, but not weakly.

Property $(\mathcal{G})$

We say that a Boolean algebra $\mathbb{B}$ satisfies property $(\mathcal{G})$, if for every normal sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}$ there is $G \in \mathbb{B}$ and pairwise disjoint sets $(H_n)_{n\in\mathbb{N}} \subseteq \mathbb{B}$ such that

- For infinitely many $n \in \mathbb{N}$
  - $|\nu_n(G \cap H_n)| \geqslant 0.3$ and
  - $|\nu_n|(H_n) \geqslant 0.9$.
- For infinitely many $n \in \mathbb{N}$
  - $G \cap H_n = \varnothing$ and
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Proposition

If $\mathbb{B}$ satisfies $(\mathcal{G})$, then it has the Grothendieck property.

Theorem (Talagrand)

Assume CH. Then there exists a balanced Boolean with the property $(\mathcal{G})$.

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Theorem (G. & Widz)

It is consistent with any possible size of $\mathfrak{c}$ that there exists a balanced algebra (of size ω_1) with the property $(\mathcal{G})$.

Sketch of the construction under CH

We construct a balanced algebra $\mathbb{B} \subseteq \mathrm{Bor}(C)$ with the property $(\mathcal{G})$ as a union

$$\mathbb{B} = \bigcup_{\alpha < \omega_1} \mathbb{B}_\alpha,$$

where $\mathbb{B}_\alpha$ are constructed by induction.

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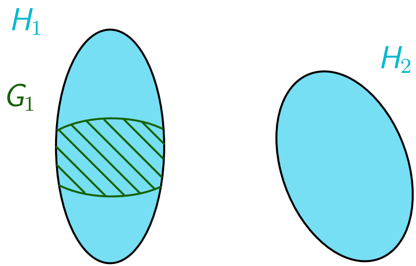
- We start with $\mathbb{B}_0 = \mathrm{Clop}(C)$
- If β is a limit ordinal, then

$$\mathbb{B}_\beta = \bigcup_{\alpha<\beta} \mathbb{B}_\alpha$$

- While constructing $\mathbb{B}_{\alpha+1}$ we are given some normal sequence $(\nu_n)_{n\in\mathbb{N}}$ of measures on $\mathbb{B}_\alpha$ and we add a new set that is a witness for the property $(\mathcal{G})$ (keeping everything balanced).

Construction of a witness for $(\mathcal{G})$



We find $n_1, n_2 \in \mathbb{N}$, disjoint sets $H_1, H_2 \in \mathbb{B}_\alpha$ and $G_1 \subseteq H_1$ such that

- $|\nu_{n_1}|(H_1), |\nu_{n_2}|(H_2) > 0.9$,
- $|\nu_{n_1}(G_1)| > 0.3$,
- other technical conditions that will allow us to continue the construction

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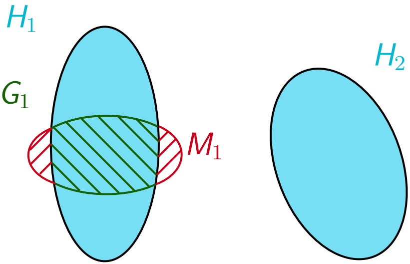

Then we find a "very small" set $M_1 \in \mathbb{B}_\alpha$ that improves the balance and $M_1 \cap (H_1 \cup H_2) = \varnothing$.

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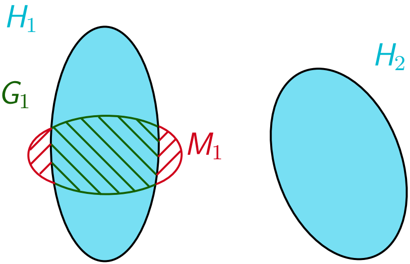

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More precisely: We are given $\varepsilon_1 > 0$ and a finite subalgebra $\mathbb{B}_1 \subseteq \mathbb{B}_\alpha$ and we want such M_1 and m_1 that

$$\mathcal{F}(\mathbb{B}_1, G_1 \cup M_1) \text{ is } (m_1, \varepsilon_1)\text{-balanced}$$

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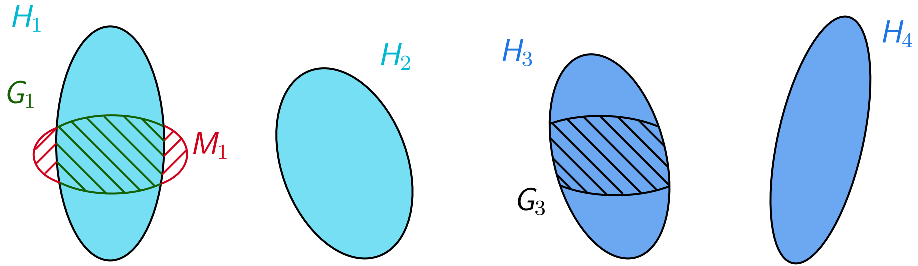

We find $n_3, n_4 \in \mathbb{N}$, disjoint sets $H_3, H_4 \in \mathbb{B}_\alpha$ and $G_3 \subseteq H_3$ such that

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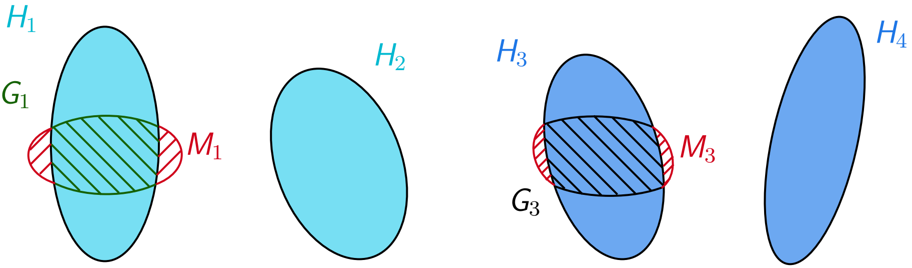

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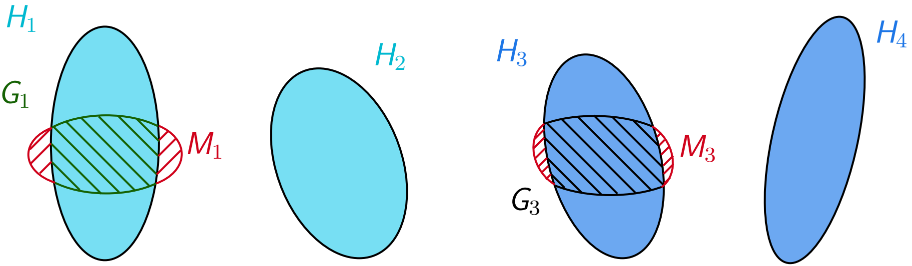

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More precisely: We are given $\varepsilon_3 > 0$ and a finite subalgebra $\mathbb{B}_1 \subseteq \mathbb{B}_3 \subseteq \mathbb{B}_\alpha$ and we want such M_3 and $m_3 > m_1$ that

$$\mathcal{F}(\mathbb{B}_i, G_1 \cup M_1 \cup G_3 \cup M_3) \text{ is } (m_i, \varepsilon_i)\text{-balanced for } i \in \{1, 3\}$$

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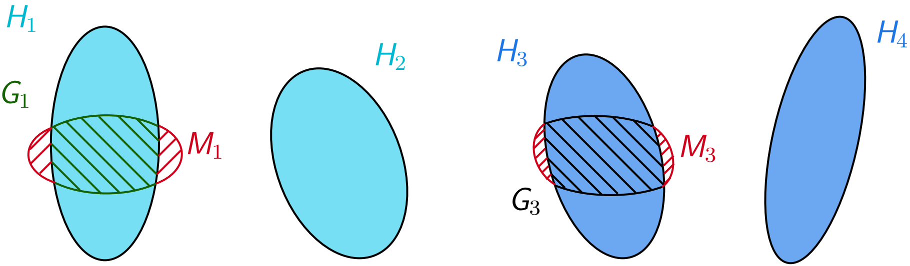

We finish taking

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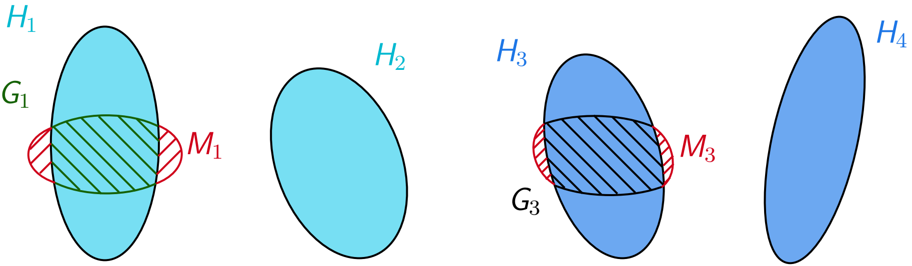

We finish taking

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Then

- $\mathbb{B}_{\alpha+1} = \mathcal{F}(\mathbb{B}_\alpha, G)$ is balanced
- G is a witness for the property $(\mathcal{G})$ for $(\nu_n)_{n \in \mathbb{N}}$

Forcing

For a countable Boolean algebra $\mathbb{B}$ we fix a representation as an increasing union of finite subalgebras:

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We define a notion of forcing $\mathbb{P}$. Conditions are of the form

$$p = (k^p, (m_n^p)_{n\leqslant k^p}, (G_n^p)_{n\leqslant k^p}, (H_n^p)_{n\leqslant k^p}, \mathcal{M}^p),$$

where

- $k^p \in \mathbb{N}$,
- $(m_n^p)_{n\leqslant k^p}$ is a strictly increasing sequence of natural numbers,
- $\mathcal{M}^p$ is a finite set of probability measures on $\mathbb{B}$ such that $\lambda \restriction \mathbb{B} \in \mathcal{M}^p$,
- $(G_n^p)_{n\leqslant k^p}$ and $(H_n^p)_{n\leqslant k^p}$ are sequences of elements of $\mathbb{B}$ such that
  - $G_n^p \cap G_l^p = H_n^p \cap H_l^p = G_n^p \cap H_l^p = \varnothing$ for $n \neq l$,
  - $\mu\left(\bigcup_{n\leqslant k^p}(G_n^p \cup H_n^p)\right) < 0.1$ for all $\mu \in \mathcal{M}^p$,
  - $\mathcal{F}\left(\mathbb{B}_n, \bigcup_{i\leqslant k^p} G_i^p\right)$ is $(m_n, 2^{-n})-$balanced for $n \leqslant k^p$.

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$$p = (k^p, (m_n^p)_{n \leqslant k^p}, (G_n^p)_{n \leqslant k^p}, (H_n^p)_{n \leqslant k^p}, \mathcal{M}^p),$$

where $q \leqslant p$, if

- $k^q \geqslant k^p$,
- $m_n^q = m_n^p$ for $n \leqslant k^p$,
- $G_n^q = G_n^p$ for $n \leqslant k^p$,
- $H_n^q = H_n^p$ for $n \leqslant k^p$,
- $\mathcal{M}^q \supseteq \mathcal{M}^p$.

Properties of $\mathbb{P}$

Let $\mathbb{G}$ be $\mathbb{P}$-generic over V. In $V[\mathbb{G}]$ we define

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Then

- the algebra $\mathbb{B}'$ generated by $\mathbb{B} \cup \{G\}$ is balanced,
- if $(\nu_n)_{n\in\mathbb{N}}$ is a normal sequence such that $(|\nu_n|)_{n\in\mathbb{N}}$ converges to a measure $\nu \in \mathcal{M}^p$ for some $p \in \mathbb{G}$, then G is a witness for the property $(\mathcal{G})$ for this sequence.

Final step

To obtain a model with a balanced algebra with the property $(\mathcal{G})$ we extend our algebras ω_1 times using finitely supported iteration of described forcings.

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In this model we have

$$\mathfrak{p} = \mathfrak{s} = \mathfrak{cov}(\mathcal{M}) = \omega_1$$


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